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Problem:

Define, for elements $x, y \in R$, R a ring, $x \mid_R y$ (read "x right-divides y in R") iff $\exists z \in RPR : y = xz$. Further, define $r \in R$ to be right-prime if when $r \mid_R xy$ for some $x, y \in R$, then $r \mid_R x$ or $r \mid_R y$. Finally, define R, a ring, to be a Right-prime Ring (RPR) if for all $x \in R$, x is right-prime. We aim to show that every Right prime ring is also a division ring.

Solution:

Let the notation: $x \in RPR$ mean $x \in R$, R a Right-prime ring. Our first aim here will be to prove the zero product property, we start with the following lemma:

Lemma **1** $\forall a \in RPR$, $0 \mid_R a \Rightarrow a = 0$

Proof: Assume we have $a \in RPR$, such that $0 \mid_R a$. By definition of right-divides, $a = 0 * x$ for some $x \in RPR$. Then $a = 0 * x = 0$, so therefore $a = 0$. ∎

With this, we can prove the Zero Product Property (ZPP), as follows:

Lemma **2** $\forall a, b \in RPR$, $ab = 0 \Rightarrow a = 0 \ or \ b = 0$

Proof: Assume we have $a, b \in RPR$ such that $ab = 0$ and assume $a \neq 0$. Certainly $0 = 0 * 0$, so substitute in $ab = 0$ to get $ab = 0 * 0$. Then, by definition of right-divides, $0 \mid_R ab$. Since R is an RPR, then $0 \mid_R a$ or $0 \mid_R b$. Since $a \neq 0$, then by the contrapositive of Lemma 1, $0 \mid_R a$ cannot hold, so $0 \mid_R b$ must hold. But then by Lemma 0, $b = 0$. ∎

Since the ZPP holds, we know that cancellation (that is, $a * x = b * x \iff a = b$) holds in RPR's. Next, we need this lemma and its corollary to prove the existence of a non-zero idempotent element.

Lemma **3** $\forall x \in RPR$, $x \mid_R x$

Proof: Let $x \in RPR$ certainly we have $xx = xx$, so by definition of right-divides $x \mid_R xx$. Since by definition every element of an RPR is right-prime, then we have that $x \mid_R x$ or $x \mid_R x$, but this is equivalent to saying $x \mid_R x$. Therefore, the lemma holds. ∎

It is trivial to show then that for every element of an RPR, there is a element which is right-neutral for that element. That is, $\forall x \in RPR, \exists r_x x = x r_x$

Now we show that every RPR has exactly one nonzero idempotent element, then we will show that this element is a multiplicative identity. As follows:

Lemma **4** $\exists! r, r \neq 0, r = r^2$

Proof: First, by lemma 3, we have x and r, such that $x = xr$, assume x is nonzero, then r must also be nonzero, multiplying both sides on the right by r will give $xr = xr^2$, we can cancel x on the left to get $r = r^2$. So r is idempotent

Finally, we want to show that r is unique. So assume $\exists s$ such that $s \neq 0$ and $s = s^2$, then consider the following:

$s^2 = s$

$s^2 - s = 0$

$rs^2 - rs = r * 0 = 0$

$(rs - r)s = 0$

So then, by the ZPP, either $s = 0$, or $rs - r = 0$, since, by assumption, $s \neq 0$, then $rs - r = 0$. This gives $rs = r$, but since $r = r * r$, we have $rs = rr$, cancel r's to get, we have that $s = r$, $\therefore r$ is unique. ■

Now with this, we can show that RPR's have identity.

Theorem **1** $\exists e \in RPR, \forall x \in RPR, xe = x = ex$

Proof: We will show that r from Lemma 4 is, in fact, the identity. So assume we have $e \neq 0$ and $e = e^2$, further, we know that this e exists and is unique. Consider the cases:

$x = xe$:
Since $e = e^2$, then certainly $xe = xe^2$, cancelling on the left gives $x = xe$, as desired.

$x = ex$:
From the above, we have $x = xe$, multiplying on the right by x we get $xx = xex$, cancelling on the left gives $x = ex$, which was what we wanted.

$xe = ex$:
Finally, since $xe = x$, and $x = ex$, then by substitution $xe = ex$.

All cases considered, the result holds. ■

Theorem **2** $\forall x \neq 0, x \in RPR, \exists y \neq 0, y \in RPR : xy = e = yx$

Proof: Consider the cases:

$x = e$:
If $x = e$, then trivially $x * e = e * e = e$.

$x \neq e$:
By lemma 3, $xx \mid_R xx$. Since $xx \in RPR$, then xx is right prime (Definition of RPR), so $xx \mid_R x$. By definition of right prime, this implies that $xxy = x$ for some $y \in RPR$. Cancel on the left to get $xy = e$.

So every element of an RPR has a right inverse.

Finally we must show that all right inverses are left inverses, consider then the following.

Let x and y be nonzero elements of an RPR, and let y be an inverse for x, then by definition, $xy = e$, multiply both sides by x to get $xyx = x$, cancel on the left to get $yx = e$. So y is also a left-inverse for x. $\therefore \forall x$ in an RPR, x has an inverse.

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By theorems 1 and 2, we know that RPR's have identity and (for all nonzero elements) inverses, so by definition, RPR's are division rings.

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