Mathematics Magazine Problem #1811

J. Fredette, Worcester State College, Worcester, MA, March 2009

PROBLEM: Given a connected graph G with vertices $v_1, v_2, \ldots, v_n$, let d_{ij} denote the distance from v_i to v_j (That is, d_{ij} is the length of the shortest path between v_i to v_j). The Wiener index $W(G)$ of G is defined by:

$$W(G) = \sum_{1 \le i < j \le n} d_{ij} \quad \text{or equivalently} \quad W(G) = \frac{1}{2}\sum_{i,j=1}^{n} d_{ij}$$

Let $S = \{0,1\}$, and $T_n = \{0, 1, 2 \ldots n\}$. Then the Grid Graph, G_n, has the set of vertices $V = \{(v_i, v_j) : v_i \in S, v_j \in T_n\}$, and the set of edges $E = \{\{(v_i, v_j), (v_{i'}, v_{j'})\} : |v_i - v_{i'}| + |v_j - v_{j'}| = 1\}$. Similarly, the Comb Graph C_n, has the set of vertices $V = \{(v_i, v_j) : v_i \in S, v_j \in T_n\}$, and the set of edges $E = \left\{ \{(v_i, v_j), (v_{i'}, v_{j'})\} : v_i - v_{i'} + |v_j - v_{j'}| = \left\{ \begin{array}{ll} 0 & \text{if } v_i = v_{i'} = 1 \\ 1 & \text{otherwise} \end{array} \right. \right\}$.
For all n, characterize $W(G_n)$ and $W(C_n)$.

PROOF: We first notice that a shortest path in the grid graph is simply the taxicab distance between vertex (x_1, y_1) and (x_2, y_2). We can write a simple algorithm with the Wiener Indices for the first few G_n and C_n,

$$\begin{array}{rclcrcl} W(G_1) & = & 1 & \quad & W(C_1) & = & 1 \\ W(G_2) & = & 8 & & W(C_2) & = & 10 \\ W(G_3) & = & 25 & & W(C_3) & = & 31 \\ W(G_4) & = & 56 & & W(C_4) & = & 68 \\ W(G_5) & = & 105 & & W(C_5) & = & 125 \end{array} \qquad (1)$$

We will use these along with the following to come up with a recursive formula for $W(G_n)$ and $W(C_n)$. We define the difference of a function $d(f(n)) = f(n+1) - f(n)$, we then take the difference of the list of $W(G_n)$ and $W(C_n)$. We now aim to show that the third difference of $W(G_n)$ and $W(C_n)$ is equal to 4. First note that expanding $d^3(f(n))$ for some f yields:

$$\begin{aligned} d^3(f(n)) &= d(d(f(n+1))) - d(d(f(n))) \\ &= d(f(n+2)) - d(f(n+1))) - [d(f(n+1)) - d(f(n))] \\ &= f(n+3) - 3f(n+2) + 3f(n+1) - f(n) \end{aligned}$$

LEMMA **1** *If $d^3(f(n)) = k$ for all finite n up to some value, then $d^3(f(n+1)) = k$ as well. In particular, $d^3(f(n)) = 4$ if $f(n)$ is either $W(G_n)$ or $W(C_n)$.*

PROOF: By way of contradiction, assume that $d^3(f(n+1)) \neq k$. We can expand

$$d^3(f(n+1)) = f(n+4) - 3f(n+3) + 3f(n+2) - f(n+1) \neq k.$$

Since $d^3(f(n)) = k$, we get $f(n+3) = k + 3f(n+2) - 3f(n+1) + f(n)$. Equivalently,

$$f(n) = k + 3f(n-1) - 3f(n-2) + f(n-3) \text{ for } n \geq 3.$$

Rearraging yields

$$f(n+4) - 3f(n+3) + 3f(n+2) - f(n+1) = k.$$

This contradicts our assumption $d^3(f(n+1)) \neq k$. $d^3(f(n)) = 4$ can then be easily proved by mathematical induction for $k = 4$ and substituting $f(n) = W(G_n)$ or $f(n) = W(C_n)$. ∎

Trivially by this lemma, we have that for every n, $W(G_n) = 4 + 3W(G_{n-1}) - 3W(G_{n-2}) + W(G_{n-3})$ and $W(C_n) = 4 + 3W(C_{n-1}) - 3W(C_{n-2}) + W(C_{n-3})$.

Since the recurrence for both $W(G_n)$ and $W(C_n)$ are the same, say the general recurrence is $f(n)$, we can eliminate the value of 4 by simply taking the next difference, yielding:

$$f(n) = 4f(n-1) - 6f(n-2) + 4f(n-3) - f(n-4)$$

Now apply the Z-transform to $f(n)$ – namely, substitute $f(n) = z$ and $f(n+k) = z^{k+1}$, This gives:

$$z = 4 - 6z^{-1} + 4z^{-2} - z^{-3}$$

Now multiply through by z^3 to get

$$z^4 = 4z^3 - 6z^2 + 4z - 1$$

We can notice that this is precisely $z^4 - 4z^3 + 6z^2 - 4z + 1 = (1-z)^4 = 0$ by the Binomial Theorem. So the equation has one root, namely 1, with multiplicity 4 and it is of binomial form. By the Z-transform, $f(n)$ has the form:

$$f(n) = A_0 + A_1 n 1^n + A_2 n^2 1^n + A_3 n^3 1^n + A_4 n^4 1^n$$

Thus

$$f(n) = A_0 + A_1 n + A_2 n^2 + A_3 n^3 + A_4 n^4$$

Now we substitute initial conditions for $W(G_n)$ to get coefficients, A_i, for $0 \leq i \leq 4$:

$$\begin{array}{ll} f(1) = 1 = A_0 + A_1 + A_2 + A_3 + A_4 & \& \\ f(2) = 8 = A_0 + 2A_1 + 4A_2 + 8A_3 + 16A_4 & \& \\ f(3) = 25 = A_0 + 3A_1 + 9A_2 + 27A_3 + 81A_4 & \& \\ f(4) = 56 = A_0 + 4A_1 + 16A_2 + 64A_3 + 256A_4 & \& \\ f(5) = 105 = A_0 + 5A_1 + 25A_2 + 125A_3 + 625A_4 & \end{array}$$

Solving this system yields:

$W(G_n) = 0 + -\frac{2}{3}n + n^2 + \frac{2}{3}n^3 + 0n^4 = \frac{n(2n-1)(n+2)}{3}$

By the same token, $W(C_n)$ is:

$W(C_n) = 0 + -\frac{5}{3}n + 2n^2 + \frac{2}{3}n^3 + 0n^4 = \frac{n(2n^2+6n-5)}{3}$

These two equations characterize the Weiner index of the Grid and Comb graphs for all n

∎